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Approximate method of solving equations for plate heat exchangers

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Abstract—Approximate solutions of heat transfer in plate heat exchangers are obtained using exponential approximations for the temperature in each stream. Numerical results obtained by the method presented have been compared with the exact analytical solutions. About 300 cases have been analysed.

INTRODUCTION

A MATHEMATICAL model of heat transfer in multi-channel, parallel-flow heat exchangers is defined by the system of linear ordinary differential equations with constant coefficients. Several authors have given the exact analytical solutions for selected cases [1–6]. Exact analytical solutions are possible if the eigenvalues and eigenvectors for the matrix of the system are known. It is not easy to find these values, especially for a high order of matrix characteristic polynomial. Therefore, approximate methods are needed to avoid these difficulties.

Settari and Venart [7] have presented a general numerical method which is an integral procedure using polynomial approximations of degree m for the temperatures in each channel of the exchanger. It proved very useful in calculating the heat transfer in exchangers with different arrangements of the inter-connections between individual channels and for different boundary conditions.

A single third degree polynomial gave good results compared with the exact solutions obtained by Mennicke [2, 3]. Settari and Venart have presented the results of calculations for plate heat exchangers with n channels ($n = 4$ and 10), and for $NTU_0 = 0.2$–0.8 and $C_h/C_c = 0.3$–5. NTU_0 is the number of transfer units for a single channel and C_h/C_c the ratio of stream capacity rates in the exchanger.

Using Settari and Venart's method, we carried out a number of calculations for plate heat exchangers. The results for different interconnections and selected configurations described by the boundary conditions were compared with the exact analytical solutions. A wide range of parameters was analysed: $NTU_0 = 0.05$–20, $C_h/C_c = 0.5$–10 and $n = 4, 8, 12, 16, 20, 24$.

Settari and Venart's approximate method proved to be very useful in predicting the temperature in plate heat exchangers when the values of NTU_0 calculated for a single channel were not too large ($NTU_0 = 0.05$–1); if, however, they exceeded 10, the method led to unacceptable errors, particularly for a small number of channels. To avoid these errors a modification of Settari and Venart's method is proposed.

Instead of polynomial approximations for the prediction of temperatures in each channel, we suggest a linear combination of exponential functions. The results do not differ from the exact analytical solutions over a wide range of parameters: $NTU_0 = 0.05$–20, $C_h/C_c = 0.1$–10, $n = 4$–40. Altogether, some 300 cases have been analysed.

AN APPROXIMATE METHOD OF SOLUTION FOR PLATE HEAT EXCHANGERS

A mathematical model of heat transfer in multi-channel, parallel-flow heat exchangers is defined by the system of linear ordinary differential equations [4]:

$$\frac{d\Theta_i}{dz} + \sum_{j=1}^{n} a_{ij}(\Theta_i - \Theta_j) = 0 \quad \text{for } i = 1, 2, \ldots, n \tag{1}$$

where

$$a_{ij} = \frac{k_{ij}F_{ij}}{C_i} \tag{2}$$

$$z = \left(\frac{x}{L}\right), \quad z \in (0, 1) \tag{3}$$

and

$$\Theta_i = \frac{t_i - t_{min}}{t_{max} - t_{min}}. \tag{4}$$

Dimensionless coordinate z is the ratio of linear coordinate x (length) to the length L of the exchanger. In a plate heat exchanger L is the length of a single channel. Equations (1) are linear if the usual assumptions are made [4]. Exact analytical solutions are known for several types of exchangers [1–5].

Settari and Venart [7] have presented an approximate method. Following the idea of the integral

NOMENCLATURE

C	flow stream heat capacity rate [W K^{-1}]
c	coefficient in polynomial and exponential approximation (formulae (5) and (9)) [dimensionless]
F, F_{ij}	surface area of the entire exchanger, surface area of heat transfer between media i and j [m^2]
F_0	surface area of one plate [m^2]
k, k_{ij}	overall heat transfer coefficient in plate heat exchanger, heat transfer coefficient between media i and j [W m^{-2} K^{-1}]
L	length of heat exchanger or length of channel in plate heat exchanger [m]
m	degree of polynomial (formula (5)) [dimensionless]
n	total number of channels
NTU	number of heat transfer units for the entire heat exchanger, $k \cdot F/C$ [dimensionless]
NTU_0	number of transfer units in single channel, $k \cdot F_0/C_0$ [dimensionless]
P	temperature effectiveness of cold fluid (formula (10)) [dimensionless]
t, T	temperature [K]
x	coordinate (length in the x-direction)
z	dimensionless coordinate, x/L.

Greek symbols

α	coefficient in exponential approximation (formula (9)) [dimensionless]
Θ	dimensionless temperature (formula (4))
$\Delta\Theta_m$	dimensionless mean temperature difference
$\Delta\Theta_{ln}$	dimensionless log-mean temperature difference
ϕ	effectiveness of exchanger [dimensionless]; $\phi_h = \Theta_{h,in} - \Theta_{h,out}$ refers to hot fluid
ψ	log-mean temperature difference correction factor (formula (11)) [dimensionless].

Subscripts

c	cold
h	hot
i, j	channels i, j
in	inlet
l	nodal points for approximating function
min	minimum
max	maximum
out	outlet
u, v	integration limits
0	single channel or one plate.

method an approximate solution to Θ_i is assumed to be a polynomial of degree m:

$$\Theta_i = \sum_{l=0}^{m} c_{il} z^l. \tag{5}$$

The coefficients of this polynomial c_{il} are determined by $(m+1)$ values of $\Theta_i^{(j)}$ for $i = 1, 2, \ldots, n$, $j = 0, 1, \ldots, m$:

$$\Theta_i^{(0)} = \Theta_i(z_0) = \sum_{l=0}^{m} c_{il} z_0^l$$

$$\Theta_i^{(1)} = \Theta_i(z_1) = \sum_{l=0}^{m} c_{il} z_1^l$$

$$\vdots$$

$$\Theta_i^{(m)} = \Theta_i(z_m) = \sum_{l=0}^{m} c_{il} z_m^l. \tag{6}$$

It is assumed that

$$0 = z_0 < z_1 < z_2 < \cdots < z_m = 1.$$

Substitution of the function (5) into the ith equation of (1) gives

$$\Theta_i(z_u) - \Theta_i(z_v) = -\sum_{j=1}^{m} \left(\int_{z_u}^{z_v} a_{ij}(\Theta_i - \Theta_j)\, dz \right)$$

$$i = 1, 2, \ldots, n. \tag{7}$$

Integrating (7) for different combinations of chosen values of $z_0, z_1, \ldots, z_m$ as the integration limits z_u and z_v we obtain $m \times n$ linear algebraic equations.

The remaining n equations are found from the boundary conditions. Having $(m+1) \times n$ such equations, it is possible to find all the coefficients c_{il} in (6) and, consequently, the temperatures $\Theta_i^{(j)}$ (for $i = 1, 2, \ldots, n; j = 0, 1, \ldots, m$).

As has been mentioned earlier, Settari and Venart's method yields accurate results when the value of NTU_0 calculated for a single channel is not too large; otherwise (NTU_0 of the order of 10 or more) the method leads to high errors, especially for small numbers of channels.

A modification of this method is thus presented using a linear combination of exponential functions for the temperatures in each channel instead of polynomial approximations. Such a function for a two-medium, plate heat exchanger has the following form:

$$\Theta_i = c_{i0} + c_{i1} \exp(\alpha_1 z) + c_{i2} \exp(-\alpha_1 z) + c_{i3} \exp(\alpha_2 z) + c_{i4} \exp(-\alpha_2 z) \quad \text{for } i = 1, 2, \ldots, n \tag{8}$$

where

$$\alpha_1 = kF_0/C_{h,0} + kF_0/C_{c,0}$$

$$\alpha_2 = kF_0/C_{h,0} - kF_0/C_{c,0}.$$

The values of α_1 and α_2 refer to a single channel, k is

Table 1. Comparison of results of calculations Θ and ϕ_h for plate heat exchangers (Fig. 1(c))

				Maximum error of Θ		Error of ϕ_h (%)		
kF/C_h	$kF_0/C_{h,0}$	$kF_0/C_{c,0}$	C_h/C_c	Settari and Venart method	Exponential method	Settari and Venart method	Exponential method	Number of channels
0.5	0.333	0.167	0.5	0.0001	0.0000	0.0054	0.0001	4
		0.333	1	0.0002	0.0000	0.0154	0.0001	
		0.667	2	0.0013	0.0000	0.0647	0.0023	
		1.667	5	0.0138	0.0011	0.6531	0.0563	
2	1.333	0.667	0.5	0.0103	0.0008	0.6459	−0.0570	
		1.333	1	0.0148	0.0014	1.5490	0.1885	
		2.667	2	0.0484	0.0080	4.6070	−0.7744	
		6.667	5	0.2159	0.0526	17.5000	−3.5364	
5	3.333	1.667	0.5	0.0746	0.0155	7.6620	−1.4890	
		3.333	1	0.0926	0.0220	13.3948	−3.1633	
		6.667	2	0.2280	0.0649	24.0853	−6.1000	
		16.667	5	—	0.1204	—	−8.7558	
0.5	0.286	0.143	0.5	0.0000	0.0000	0.0046	0.0000	8
		0.286	1	0.0002	0.0000	0.1283	−0.0003	
		0.571	2	0.0010	0.0000	0.0545	−0.0018	
		1.429	5	0.0099	0.0008	0.5597	−0.0474	
2	1.143	0.571	0.5	0.0080	0.0007	0.5578	−0.0505	
		1.143	1	0.0125	0.0014	1.3641	−0.1747	
		2.286	2	0.0394	0.0068	4.2147	−0.7535	
		5.714	5	0.1763	0.0444	17.3227	−4.0249	
5	2.857	1.429	0.5	0.0590	0.0135	7.1824	−1.5028	
		2.857	1	0.0790	0.0221	12.9173	−3.3158	
		5.714	2	0.1941	0.0601	24.1662	−6.9171	
		14.286	5	—	0.1198	—	−10.9508	
0.5	0.263	0.132	0.5	0.0000	0.0000	0.0040	0.0001	20
		0.263	1	0.0001	0.0000	0.0112	0.0001	
		0.526	2	0.0007	0.0000	0.0472	−0.0016	
		1.319	5	0.0079	0.0006	0.4965	−0.0400	
2	1.053	0.526	0.5	0.0063	0.0005	0.4961	−0.0431	
		1.053	1	0.0102	0.0011	1.2313	−0.1542	
		2.105	2	0.0337	0.0052	3.9137	−0.6940	
		5.263	5	0.1549	0.0379	17.0475	−4.0972	
5	2.632	1.316	0.5	0.0503	0.0106	6.7856	−1.4223	
		2.632	1	0.0678	0.0182	12.4359	−3.2388	
		5.263	2	0.1745	0.0529	23.8950	−7.1258	
		13.158	5	—	0.1151	—	−12.3391	

—, the function exceeds the range of dimensionless temperature.

the overall heat transfer coefficient assumed constant in each channel as well as over the whole exchanger, F_0 represents the surface area of one plate, and $C_{h,0}$ and $C_{c,0}$ are the heat capacities of the hot and cold medium in a single channel.

Following Settari and Venart, the coefficients of equations (8) are calculated by substituting (8) into the ith equation of (7). Integrating (7) for four different combinations of the values of $0 = z_0 < z_1 < z_2 < z_3 < z_4 = 1$ as the integration limits we obtain $4n$ linear algebraic equations. In our equations $z_0 = 0$, $z_1 = \frac{1}{4}$, $z_2 = \frac{2}{4}$, $z_3 = \frac{3}{4}$, $z_4 = 1$. The missing n equations are provided by the boundary conditions.

Having $5n$ such equations we can find all unknown coefficients in (8), and hence all unknown temperatures in the exchanger.

For $kF_0/C_{h,0} = kF_0/C_{c,0}$, we use the function (9) instead of (8)

$$\Theta_i = c_{i0} + c_{i1}\exp(\alpha z) + c_{i2}\exp(-\alpha z) + c_{i3}z \quad (9)$$

where

$$\alpha = kF_0/C_{h,0} + kF_0/C_{c,0}.$$

The number of unknown coefficients is $4n$. A total of $3n$ equations may be obtained upon the substitution of (9) into (7). The remaining n equations are again found from the boundary conditions.

The method proposed has been termed 'exponential'. The form of the function (8) or (9) results from a particular structure of the flow in the plate exchangers (both co-current and countercurrent flow). The coefficients α_1, α_2 or α are the eigenvalues of a system of differential equations. This system is the mathematical model of an exchanger formed by two adjacent channels in a plate exchanger.

COMPARISON AND ANALYSIS OF THE RESULTS

Calculations of the temperature distribution and thermal effectiveness have been carried out for a plate heat exchanger with a configuration of flow as in Fig. 1. For such interconnections between the channels we may obtain exact analytical solutions (with eigenvalues taken from the literature [4]). A wide range of parameters ($NTU_0 = 0.05$–20 in a single channel, ratio of total heat capacity rates $C_h/C_c = 0.2$–5, number of channels $n = 4, 8, 16, 20, 24$) has been investigated. Temperature distribution and thermal effectiveness of the hot fluid are calculated analytically and by means of approximate methods (Settari and Venart's and exponential). Maximum error and mean square error of the temperature distribution are determined for each variant. Also, the relative error concerning the thermal effectiveness is calculated.

It has been found that for low values of NTU_0 in a single channel (0.05–1) the two approximate methods are sufficiently accurate. The error concerning thermal effectiveness was less than 0.6% for Settari and Venart; for the exponential method it did not exceed 0.06%. However, for exchangers with high values of NTU_0 and low number of channels, Settari and Venart's method leads to unacceptable errors above 25%

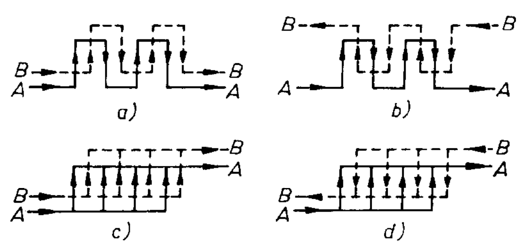

FIG. 1. Scheme of flow arrangements in a plate heat exchanger.

as concerns relative thermal effectiveness, compared with the analytical values. In several cases this method could not be used at all. For all these cases the error of the exponential method did not exceed 13%. The temperature distribution in individual channels of a plate heat exchanger is shown in Figs. 2 and 3.

A variant has been chosen which leads to large discrepancies between the analytical solution and the two approximate solutions. Table 1 presents maximum errors in the temperature distribution and relative errors concerning the effectiveness of the hot medium, ϕ_h, for the exchanger shown in Fig. 1(c). All errors are calculated relative to the values found analytically for selected parameters.

To test the approximate exponential method calculations of plate exchangers have been carried out for about 300 cases. Flow arrangements and interconnections between channels were determined by:

- number of passes of both fluids;
- number of channels in one pass;
- boundary conditions.

The results have been compared with those presented by Kandlikar and Shah [8] for various interconnections between the channels (described by the number of passes) and for various configurations of the exchangers. A wide range of NTU, C_h/C_c and n has been studied.

To calculate the temperature distribution, Kandlikar and Shah divided each channel into 100 sections. With difference equations written for each section in individual channels they solved simultaneously a system of equations using the Gauss–Seidel iterative finite difference method. The total number of iterations varied from 15 to 50 depending on the arrangement of passes and the total number of plates. The thermal effectiveness was obtained with an accuracy of about 0.0001. It may be noted that the exponential method presented in this paper requires only the solution of $5n$ linear algebraic equations (where n is the total number of channels). The results

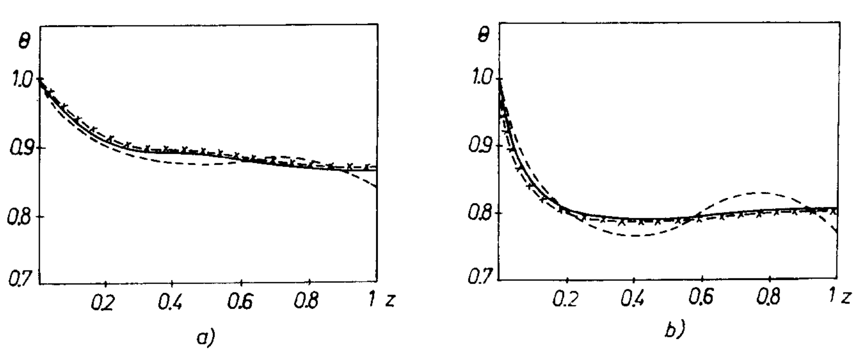

FIG. 2. Comparison of temperature profiles in plate exchanger ($kF_0/C_{h,0} = 1.3333$, $kF_0/C_{c,0} = 6.6667$, $n = 4$, flow as in Fig. 1(c)). (a) Channel 1. (b) Channel 3. (——) Exact solution; (----) Settari and Venart method; (– × – × –) exponential method.

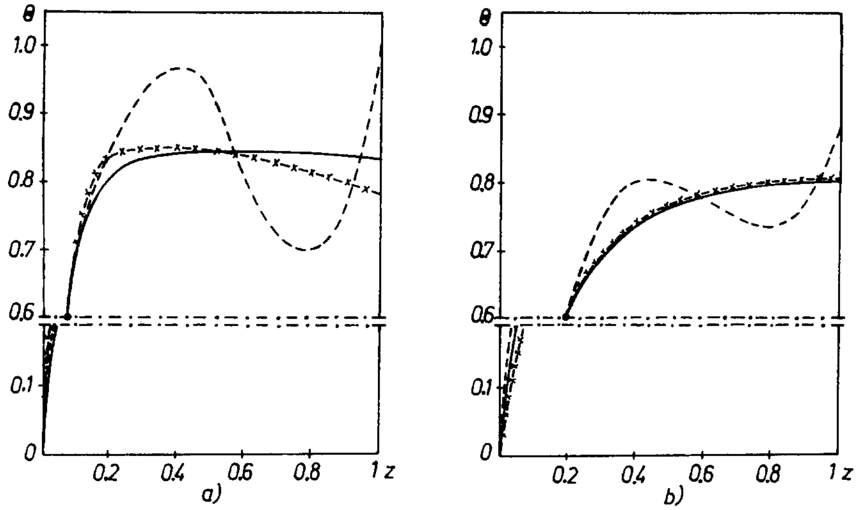

FIG. 3. Comparison of temperature profiles in plate exchanger ($kF_0/C_{h,0} = 1.3333, kF_0/C_{c,0} = 6.6667, n = 4$, flow as in Fig. 1(c)). (a) Channel 2. (b) Channel 4. (——) Exact solution; (----) Settari and Venart method; (– × – × –) exponential method.

obtained using the exponential method agree surprisingly well with those of Kandlikar and Shah [8].

The outlet temperatures of the cold fluid in each variant were compared as

$$P_c = \frac{T_{c,out} - T_{c,in}}{T_{h,in} - T_{c,in}} = \Delta\Theta_c. \quad (10)$$

In all cases, the error was less than 0.0001 for the given range of thermal parameters. The log-mean temperature difference correction factors ψ were also compared

$$\psi = \frac{\Delta\Theta_m}{\Delta\Theta_{ln}}. \quad (11)$$

In the above formula $\Delta\Theta_m$ is the dimensionless mean temperature difference, and $\Delta\Theta_{ln}$ the dimensionless log-mean temperature difference.

CONCLUSIONS

A method of calculating the temperature distribution and thermal efficiency in plate exchangers has been proposed, based on an approximation of the temperature of a medium in individual channels by means of linear exponential functions (equations (8) and (9)). The method leads to correct results over a wide range of thermal parameters and for various interconnections of the channels in the exchanger.

An advantage of the exponential method lies in the fact that in order to obtain highly accurate results it is sufficient to use a personal computer of the IBM PC type. It also seems that the method proposed may be successfully applied to other types of multichannel, parallel-flow heat exchangers.

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METHODE APPROCHEE POUR RESOUDRE LES EQUATIONS RELATIVES AUX ECHANGEURS THERMIQUES A PLAQUES

Résumé—Des solutions approchées de transfert thermique dans les échangeurs à plaques sont obtenues en utilisant des approximations exponentielles pour les températures dans chaque écoulement. Des résultats numériques obtenus par la méthode présentée ont été comparés avec les solutions analytiques exactes. Environ 300 cas ont été analysés.

NÄHERUNGSVERFAHREN ZUR LÖSUNG DER GLEICHUNGEN FÜR EINEN PLATTENWÄRMEAUSTAUSCHER

Zusammenfassung—Mit Hilfe eines exponentiellen Näherungsverfahrens für die Temperaturberechnung in beiden Massenströmen eines Plattenwärmeaustauschers konnte eine Näherungslösung für den Wärmeaustausch gefunden werden. Die mit diesem Verfahren numerisch berechneten Ergebnisse werden mit exakten analytischen Lösungen verglichen. Ungefähr 300 verschiedene Fälle werden dabei analysiert.

ПРИБЛИЖЕННЫЙ МЕТОД РЕШЕНИЯ УРАВНЕНИЙ ДЛЯ ПЛАСТИНЧАТЫХ ТЕПЛООБМЕННИКОВ

Аннотация—С использованием экспоненциальных приближений для температур в потоке получены приближенные решения задач теплопереноса в пластинчатых теплообменниках. Проведено сравнение численных результатов с точными аналитическими решениями. Проанализировано около 300 случаев.